# Waveform Relaxation for the Computational Homogenization of Multiscale Magnetoquasistatic Problems

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**This paper deals with the application of the waveform relaxation method for the homogenization of multiscale magnetoquasistatic problems. In the proposed approach, the macroscale problem and the mesoscale problems are solved separately using the finite element method on the entire time interval for each waveform relaxation iteration. The exchange of information between both problems is carried out using the heterogeneous multiscale method.**

***Index Terms*—Multiscale modeling, Homogenization, Waveform relaxation method, Cosimulation method, FE method, Eddy currents, Magnetoquasistatic problems.**

## I. Introduction

THE recent use of the heterogeneous multiscale methods (HMM [1]) has allowed to accurately solve magnetoquasistatic (MQS) problems involving hysteretic materials [2], [3]. The method requires the solution of one macroscale and many mesoscale problems in a coupled formulation based on the Finite Element (FE) method: each mesoscale problem is solved for recovering the missing macroscale constitutive law at one Gauss point of the macroscale problem. In [2], [3] the coupled problem was solved using a classical Newton-Raphson scheme with equal time step sizes at both scales. However, the use of different time steps becomes important for problems involving different dynamics at both scales. In the case of soft ferrite studied in [4], it was indeed shown that capacitive effects occurring at the mesoscale could be accounted for by upscaling proper homogenized quantities.

In this paper we propose a novel approach that provides a natural setting for the use of different time steps. The approach applies the waveform relaxation method [5], [6] to the homogenization of MQS problems: the macroscale problem and the mesoscale problems are solved separately on time intervals and their time-dependent solutions are exchanged in a fixed point iteration.

## II. The classical HMM method

The HMM method for the MQS problems is explained in detail in [2], [3]. One macroproblem accounting for the slow variations of the finescale solution and many mesoproblems used for computing missing information at Gauss points are solved iteratively at each time step of a integration scheme such as backward Euler. In the following, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{h}, \boldsymbol{j}, \sigma$ denote the magnetic vector potential, the magnetic flux density, the magnetic field strength, the electric current density and the conductivity, respectively. Indices $\mathrm{M}, \mathrm{m}$ and care used for denoting the macroscale, the mesoscale and correction terms, respectively whereas $\Omega, \Omega_{\mathrm{c}}$ and $\Omega_{\mathrm{s}}$ denote the total, the conducting and inductor domains, respectively.

### A. The macroscale problem

The macroscale weak form of the problem in $\boldsymbol{a}$-formulation reads: find $\boldsymbol{a}_{\mathrm{M}} \in \boldsymbol{H}_e(\mathrm{curl};\Omega)$ such that [3]

$$\Big(\boldsymbol{\mathcal{H}}_{\mathrm{M}}(\boldsymbol{b}_{\mathrm{c}}+\boldsymbol{b}_{\mathrm{M}}), \mathrm{curl}_x\, \boldsymbol{a}_{\mathrm{M}}^{'}\Big)_{\Omega} + \Big(\sigma_{\mathrm{M}}\partial_t \boldsymbol{a}_{\mathrm{M}}, \boldsymbol{a}_{\mathrm{M}}^{'}\Big)_{\Omega_{\mathrm{c}}} + \Big\langle \boldsymbol{n}\times\boldsymbol{h}_{\mathrm{M}}, \boldsymbol{a}_{\mathrm{M}}^{'}\Big\rangle_{\Gamma} = \Big(\boldsymbol{j}_{\mathrm{s}}, \boldsymbol{a}_{\mathrm{M}}^{'}\Big)_{\Omega_{\mathrm{s}}}, \quad (1)$$

for all $\boldsymbol{a}_{\mathrm{M}}^{'} \in \boldsymbol{H}_e^0(\mathrm{curl};\Omega)$ and for all $t \in \mathcal{I} = [t_0, t_{\mathrm{end}}]$, and where $\boldsymbol{H}_e^0(\mathrm{curl};\Omega)$ is the appropriate function space for the vector potential with boundary conditions. The missing constitutive law $\boldsymbol{\mathcal{H}}_{\mathrm{M}}(\boldsymbol{b}_{\mathrm{c}}+\boldsymbol{b}_{\mathrm{M}})$ is computed using the macroscale $\boldsymbol{b}_{\mathrm{M}} = \mathrm{curl}_x\, \boldsymbol{a}_{\mathrm{M}}$ and mesoscale magnetic fluxes $\boldsymbol{b}_{\mathrm{c}} = \mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{c}}$.The latter is computed by solving mesoscale problems on cells around Gauss points as explained in the next section.

### B. Mesoscale problems

The mesoscale weak form reads: find the mesoscale correction $\boldsymbol{a}_{\mathrm{c}} \in \boldsymbol{H}_e(\mathrm{curl};\mathcal{Y})$ ($\mathcal{Y}$ denoting the cell $\Omega_{\mathrm{m}}$ with periodic boundary conditions [7]) such that [3]

$$\Big(\boldsymbol{\mathcal{H}}(\mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{c}}+\boldsymbol{b}_{\mathrm{M}}), \mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{m}}} + \Big(\sigma\partial_t \boldsymbol{a}_{\mathrm{c}}, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{mc}}} = \Big(\sigma(\boldsymbol{e}_{\mathrm{M}} - \kappa(\partial_t \boldsymbol{b}_{\mathrm{M}}\times\boldsymbol{y})), \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{mc}}}, \quad (2)$$

for all $\boldsymbol{a}_{\mathrm{c}}^{'} \in \boldsymbol{H}_e(\mathrm{curl};\mathcal{Y})$ and for all $t$ in the macroscale time interval $[t_i, t_{i+1}]$. Using a more compact notation, equation (2) becomes: find $\boldsymbol{a}_{\mathrm{c}}$ such that

$$\Big(\boldsymbol{\mathcal{H}}(\mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{c}} + \boldsymbol{M}_1\boldsymbol{f}_{\mathrm{M}}), \mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{m}}} - \Big(\sigma\partial_t \boldsymbol{a}_{\mathrm{c}}, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{mc}}} = \Big(\sigma\boldsymbol{M}_2\boldsymbol{f}_{\mathrm{M}}, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{mc}}}, \quad (3)$$

where we combined all relevant quantities in a macroscale vector source $\boldsymbol{f}_{\mathrm{M}} = (\boldsymbol{a}_{\mathrm{M}}, \boldsymbol{b}_{\mathrm{M}}, \boldsymbol{e}_{\mathrm{M}}, \partial_t \boldsymbol{b}_{\mathrm{M}}\times\boldsymbol{y})^T$. The selection matrices $\boldsymbol{M}_1,\ \boldsymbol{M}_2 \in [0,1]^{3\times 12}$ allow to recover adequate

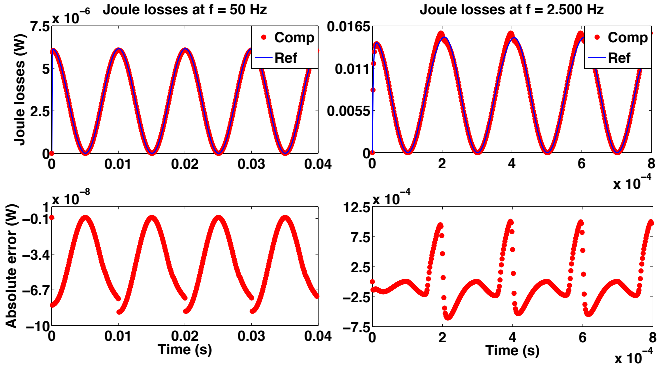


Fig. 1: Instantaneous Joule losses and absolute error between the reference ('Ref') and the computational ('Comp') solutions. Hysteretic case at $f = 50$ Hz (left) and $f = 2500$ Hz (right).

macroscale source fields from the vector $\boldsymbol{f}_{\mathrm{M}}$ for each term in (2). For instance $\boldsymbol{M}_1$ is given by:

$$\boldsymbol{M}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \end{pmatrix}$$

and the first source term in (2) is recovered as $\boldsymbol{b}_{\mathrm{M}} = \boldsymbol{M}_1 \boldsymbol{f}_{\mathrm{M}}$.

Figure 1 depicts the evolution of Joule losses for excitations at 50 Hz and 2500 Hz (which correspond to the case with pronounced skin effect) obtained in [3] using the approach described in Section II. A good agreement between Joules losses was observed for both frequencies.

## III. The waveform relaxation-based method

We employ a waveform relaxation-based approach with windowing [5]. Weak forms similar to (1) for the macroscale and (2) for the mesoscale problem are solved on a series of time windows $T_j = [t_{j-1}, t_j] \subset \mathcal{I}$ ($j = 1, 2, \dots, n$). On each time window macroscale and mesoscale problems are solved separately in time-domain (possibly using integrators with different time step sizes), such that waveforms, e.g., $\boldsymbol{a}_{\mathrm{M}}(t)$, are obtained. Afterwards the coupling between the problems is introduced by exchanging the waveforms, and solving the system iteratively. In each iteration $k$ the solutions of the macroscale, e.g. $\boldsymbol{f}_{\mathrm{M}}^{(k)}(t)$ and mesoscale problems, e.g. $\boldsymbol{a}_{\mathrm{c}}^{(k)}(t)$, are computed until convergence is reached, e.g. $\|\boldsymbol{a}_{\mathrm{c}}^{(k-1)}(t) - \boldsymbol{a}_{\mathrm{c}}^{(k)}(t)\| <$ tol.

### A. Mesoscale problems

Starting from the mesoscale weak form (3), the following mesoscale weak form is derived for each $k$-th waveform iteration: find $\boldsymbol{a}_{\mathrm{c}}^{(k)} \in \boldsymbol{H}_e(\mathrm{curl}; \mathcal{Y})$ such that

$$\Big(\boldsymbol{\mathcal{H}}(\mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{c}}^{(k)} + M_1 \boldsymbol{f}_{\mathrm{M}}^{(k-1)}), \mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{m}}} - \Big(\sigma \partial_t \boldsymbol{a}_{\mathrm{c}}^{(k)}, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{mc}}} = \Big(\sigma \mathbf{M}_2 \boldsymbol{f}_{\mathrm{M}}^{(k-1)}, \boldsymbol{a}_{\mathrm{c}}^{'}\Big)_{\Omega_{\mathrm{mc}}}, \quad (4)$$

for all $\boldsymbol{a}_{\mathrm{c}}^{'} \in \boldsymbol{H}_e(\mathrm{curl}; \mathcal{Y})$. For each time instant $t$, mesoscale fields are then used for upscaling the macroscale magnetic constitutive law $\boldsymbol{\mathcal{H}}_{\mathrm{M}}^{(k)}(t, \boldsymbol{b}_{\mathrm{M}}) := \boldsymbol{\mathcal{H}}_{\mathrm{M}}(\boldsymbol{b}_{\mathrm{c}}^{k}(t) + \boldsymbol{b}_{\mathrm{M}})$.

### B. The macroscale problem

Using the constitutive law computed from the mesoscale solutions, it is possible to solve the following macroscale weak form for the $k$-th waveform iteration: find $\boldsymbol{a}_{\mathrm{M}}^{(k)} \in \boldsymbol{H}_e(\mathrm{curl}; \Omega)$ such that

$$\Big(\boldsymbol{\mathcal{H}}_{\mathrm{M}}^{(k)}(t, \boldsymbol{b}_{\mathrm{M}}), \mathrm{curl}_y\, \boldsymbol{a}_{\mathrm{M}}^{'}\Big)_{\Omega} + \Big(\sigma \partial_t \boldsymbol{a}_{\mathrm{M}}^{(k)}, \boldsymbol{a}_{\mathrm{M}}^{'}\Big)_{\Omega_{\mathrm{c}}} = \Big\langle \boldsymbol{n} \times \boldsymbol{h}_{\mathrm{M}}^{(k)}, \boldsymbol{a}_{\mathrm{M}}^{'}\Big\rangle_{\Gamma_{\mathrm{h}}} + \Big(\boldsymbol{j}_{\mathrm{s}}, \boldsymbol{a}_{\mathrm{M}}^{'}\Big)_{\Omega_{\mathrm{s}}}, \quad (5)$$

for all $\boldsymbol{a}_{\mathrm{M}}^{'} \in \boldsymbol{H}_e^0(\mathrm{curl}; \Omega)$.

The solution of the entire multiscale problem is obtained as follows: the mesoscale solutions are initialized as $\boldsymbol{a}_{\mathrm{c}}^{(0)} = \mathbf{0}$ and thus $\boldsymbol{b}_{\mathrm{c}}^{(0)} = \mathbf{0}$. The macroscale problem can then be solved for the first iteration, i.e., one obtains $\boldsymbol{a}_{\mathrm{M}}^{(1)}$ and $\boldsymbol{f}_{\mathrm{M}}^{(1)}$. This is then used for successive waveform relaxation iterations

$$\boldsymbol{a}_{\mathrm{c}}^{(0)} \rightarrow \boldsymbol{f}_{\mathrm{M}}^{(1)} \rightarrow \boldsymbol{a}_{\mathrm{c}}^{(1)} \rightarrow \boldsymbol{f}_{\mathrm{M}}^{(2)} \rightarrow \dots \rightarrow \boldsymbol{a}_{\mathrm{c}}^{(N-1)}$$

and finally $\boldsymbol{a}_{\mathrm{M}}^{(N)}$. In addition to the flexible use of different FE bases and meshes at both scales, this approach also provides a natural setting for the use of different integrators and time step sizes. Communication costs can also be reduced in the case of parallel computations. As a drawback, the number of iterations for solving both the macroscale and the mesoscale problems may increase.

## IV. Outlook

In the extended paper, we will discuss the application of the waveform relaxation-based method in more detail. A study of the time discretization at mesoscale and macroscale will be carried out, convergence will be analyzed and the computational efficiency discussed.


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## References


[1] W. E and B. Engquist, *The heterogeneous multiscale methods*, Comm. Math. Sci., 1, no. 1, 87–132, 2003.

[2] I. Niyonzima, R.V. Sabariego, P. Dular, F. Henrotte and C. Geuzaine, *Computational homogenization for laminated ferromagnetic cores in magnetodynamics*, IEEE Trans. Magn., 49(5), 2049–2052, 2012.

[3] I. Niyonzima, *Multiscale Finite Element Modeling of Nonlinear Quasistatic Electromagnetic Problems*, Ph.D. thesis, Université de Liège, Belgium, 2014.

[4] O. Bottauscio and A. Manzin, *Comparison of multiscale models for eddy current computation in granular magnetic materials*, Journal of Comp. Phys., 253, 1–17, 2013.

[5] Jacob K. White, F. Odeh, Alberto L. Sangiovanni-Vincentelli, and Albert E. Ruehli. *Waveform relaxation: Theory and practice*, Technical Report UCB/ERL M85/65, University of California, Berkley (1985).

[6] S. Schöps, H. De Gersem and A. Bartel, *A cosimulation framework for multirate time integration of field/circuit coupled problems*, IEEE Trans. Magn., 46(8), 3233–3236, 2010.

[7] A. Visintin, *Electromagnetic processes in doubly-nonlinear composites*, Comm. Partial Differential Equations, 33, 804–841, 2008.